AMERICAN SOCIETY OF CIVIL ENGINEERS.

INSTITUTED 1852.

TRANSACTIONS.

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No. 781.

THE TRANSVERSE STRENGTH OF BEAMS AS A DIRECT FUNCTION OF THE TENSILE AND CRUSHING STRESSES OF MATERIAL.

By M. LEWINSON, M. Am. Soc. C. E.

PRESENTED APRIL 1ST, 1896.

WITH DISCUSSION.

The theory of the transverse strength of beams of any shape is based on the assumption that the intensity of fiber resistances is directly proportional to the respective distances of the fibers from an axis called neutral, and that the sum of all the fiber stresses above the axis is equal to the sum of those below. Hence the neutral axis must pass through the center of gravity of the beam section, and, as a consequence, the maximum fiber stress in beams of unsymmetrical section is at those extreme fibers which are most distant from the neutral axis. A further consequence of this theory is that no matter what

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may be the material and section of the beam, its transverse strength in regard to the neutral axis is the same, whether the extreme fibers parallel to this axis are subjected to the one or to the opposite kind of forces. It is also maintained that beams of a material whose resistances to crushing and rupture are not equal must invariably break first at the weakest fibers.

It is well known how unsatisfactory are the results of the above theory, which are substantiated by experiments in only a few cases. It is known, for instance, that cast-iron beams of angle or T section are stronger when the compression forces are acting in the webs instead of in the flanges, and that in breaking timber beams the fibers in tension rupture before the compression fibers are affected, although the tensile strength of the wood may be four or five times as great as its compressive strength. This last phenomenon, as also some experiments made on beams of different materials, leads the author to the conclusion that the lower and upper fibers of beams under transverse breaking load are strained to their maximum fiber resistances. This is the fundamental principle on which the following theory is based.

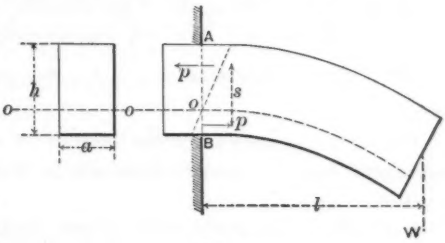


FIG. 1.

Let Fig. 1 represent a rectangular beam fixed at one end and bent, say to the breaking point, by a force W attached at the other end. The weakest place of such a beam is at the plane $A\,B$, which, under the influence of the moment $W\,l$, will rotate around a certain axis O. This rotation is opposed by $p\,s$, consisting of a leverage s and the sum of all the fiber resistances p, above or below the axis O, along the plane $A\,B$. The question arises what is the intensity of p, and what is the relation and position of s to the height of the beam section and its rotation axis respectively? Let x denote the distance of the rotation axis from the lower compressed fibers, the distance of the tension fibers will be $h - x$. The fibers at A are strained to their utmost capacity, say with a tensile stress T pounds per square inch, and the fibers at B are also strained to their utmost capacity with a crushing stress of C

pounds per square inch. These forces, being maximum at the extreme fibers, diminish in intensity toward each other in a direct ratio to the distances of the fibers from the rotation axis, where they meet and are zero. Therefore they can be represented graphically as areas of triangles, the bases of which are $T\,a$ and $C\,a$, and the heights x and $h - x$ respectively. The sum of all the forces represented by each of the triangles being the forces of a couple, they must be equal to each other, i. e.,

$$\tfrac{1}{2}\,C\,a\,x = \tfrac{1}{2}\,T\,a\,(h - x).$$

From this equation it will be found that:

$$x = \frac{T}{C+T}\,.\,h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

the fundamental expression for the position of neutral axis for rectangular beams. Making $C = T$ in the above formula, as is nearly true in wrought iron, x will equal $\frac{1}{2}\,h$ and coincide with the axis through the center of gravity. For cast iron (tension, 20 000; compression, 100 000), $\frac{C}{T} = 5$; therefore $x = \frac{1}{6}\,h$. For yellow pine, $\frac{C}{T} = \frac{1}{2}$ and $x = \frac{2}{3}\,h$.

The foregoing shows that the position of the neutral axis in a rectangular beam is nearer to those extreme fibers whose resistance is greatest.

It is further evident that the resultant of all the fiber resistances will act at the center of gravity of the triangle representing them, i. e., at one-third of the height of the triangle, or at two-thirds of the distance from the apex O; therefore $s = \frac{2}{3}\,x + \frac{2}{3}\,(h - x) = \frac{2}{3}\,h$, or the leverage of a couple in a rectangular beam is always two-thirds of its height, and its position in the section of the beam is dependent on the position of the neutral axis. This proposition is identical with the general mechanical law that the resultant fiber resistances act at the antipoles of the neutral axis in regard to the ellipse of gyration in the parts of the beam section above and below the axis. As will be seen later this general mechanical law is also true in this new theory for all sections, except that the neutral axis does not always coincide with the axis through the center of gravity.

Having found the magnitude and position of p and s, it is now possible to calculate the transverse strength of a beam, knowing only the crushing and tensile resistances of any material.

Referring again to Fig. 1:

$$p = \tfrac{1}{2}\, C\, a\, x = \tfrac{1}{2}\, T\, a\, (h - x)$$

and substituting in this equation the value for

$$x = \frac{T}{C+T}\, h,$$

it will be found: $$p = \tfrac{1}{2}\, \frac{C\,T}{C+T}\, a\, h$$

and the moment will be:

$$M = p\, s = \tfrac{1}{2}\, \frac{T\,C}{C+T}\, a\, h^2 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

For $C = T$ the moment will be $M = \frac{1}{4}\, C\, a\, h^2 = p\, s$ the familiar formula, in which $\frac{1}{4}\, a\, h^2$ is the moment of resistance of a rectangular beam by the common theory. For glass* it has been found that $C = 30\,150$ lbs., and $T = 2\,560$ lbs., and a 1-in. square bar 12 ins. between supports broke with a center load of 262 lbs. Substituting the values for C and T in formula (2), and noting that $h = a = 1$ in. and $l = 12$ ins., $3\, W\, l = 786.5$, or the theoretical center breaking weight $W = 262\frac{1}{2}$ lbs., which is identical with the one found by experiment.

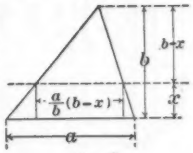


Fig. 2.

For cast iron the author made a series of experiments on beams of various cross-sections, the results of which are appended in Table No. 1. Before explaining these it is desirable to show the derivation from this theory of formulas for other than rectangular shapes.

Let Fig. 2 represent a section of a triangular beam, having a base a, and height b. At the apex the fibers are compressed with a crushing stress C per square inch, and the fibers at the base are stretched with a tensile stress T per square inch. If the neutral axis passes through the triangle parallel to and at distance x from the base, the sides of the triangle will cut from the neutral axis a distance $\frac{a}{b}(b - x)$, and the difference between a and $\frac{a}{b}(b - x)$ will be $\frac{a}{b}\, x$. Knowing also that the sum of all the intensities in a triangle in regard to its base is

* See "Strength of Materials," by Thomas Box.

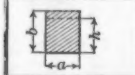
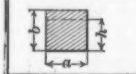
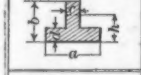
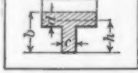
I	II	III				IV	V		VI	VII	VIII		IX
Section of Beam		DIMENSIONS				Clear Span	Ultimate Strength per. sq. inch		Distance of neutral Axis h	Leverage of Couple	Center Breaking Load in Pounds		Remarks
		a	b	c	d		Tensile	Crushing			By Test	By Rule	
	I	1.011"	2.02"			5'0"	19620	81697	1.6288"	1.34667"	1316	1450	Broke in center
	II	0.99	2.00			"	18772	79491	1.6179	1.33333	1165	1323	Broke $2\frac{3}{4}$" from cen.
	III	1.012	2.02			"	21560	83006	1.6035	1.34667	1446	1585	Broke in center
	I	1.836	1.862			4'0"	17676	75954	1.51048	1.24133	2400	2535	Broke $1\frac{1}{2}$" from cen.
	II	1.860	1.865			"	17270	75590	1.51466	1.25333	2321	2519	Broke $\frac{1}{4}$" from cen.
	III	1.865	1.861			"	17538	74586	1.50672	1.24067	2501	2429	Broke $\frac{1}{8}$" from cen.
	I	2.02	2.04			5'0"	21680	81828	0.91612	1.19958	1346	1364	Broke $1\frac{1}{2}$" from cen.
	II	2.03	2.03			"	22308	77256	0.87962	1.18639	1386	1348	Broke $\frac{1}{4}$" from cen.
	III	2.04	2.09			"	25033	90451	0.92188	1.22513	1725	1640	Broke in 2 places $4\frac{1}{2}$ from center
	I	2.00	2.05			"	18430	75532	0.14717	1.05013	665	760	Broke $1\frac{3}{8}$" from cen.
	II	2.04	2.06			"	22944	89678	0.15421	1.05636	826	969	Broke $2\frac{13}{16}$ from cen.
	III	2.03	2.05			"	23204	88253	0.15719	1.05188	860	962	Broke $\frac{1}{4}$" from cen.
	I	4.06	3.00	0.425	0.350	"	27555	115633	1.413260	2.2309	4770	5822	Broke $\frac{1}{4}$" from cen.
	II	4.115	3.05	0.455	0.370	"	26723	98806	1.312192	2.2422	4520	5839	Broke in center
	III	4.155	3.05	0.430	0.360	"	27041	110475	1.383354	2.2643	4990	5975	Broke in center
	I	4.160	3.10	0.440	0.400	"	25816	115498	3.0217	1.9466	2400	2441	Broke $\frac{3}{8}$" from cen.
	II	4.160	3.05	0.420	0.365	"	23841	109616	2.9789	1.8906	2220	2042	Broke in center
	III	4.150	3.04	0.420	0.370	"	21420	102243	2.97124	1.87753	2080	1826	Broke $\frac{3}{4}$" from cen.

TABLE No. 1.—RESULTS OF EXPERIMENTS ON BEAMS OF VARIOUS CROSS-SECTIONS.

one-third of the area of the triangle multiplied by the unit of resistance, the following equation may be written :

$$\frac{1}{6}\, C\, \frac{a}{b}\, (b - x)^2 = \frac{1}{2}\, T\, a\, x - \frac{1}{6}\, T\, \frac{a}{b} x^2 \ldots\ldots\ldots\ldots\ldots (3)$$

From which $x = \dfrac{b}{2\,(C + T)}\, [2\,C + 3\,T \pm \sqrt{(8\,C + 9\,T)\,T}] \ldots\ldots (4)$

If the bar is reversed so that the crushing stress C acts at the base and the tensile stress T at the apex of the triangle, it is only necessary to substitute in formula (4) T for C, and C for T, to get the value for x, which is :

$$x = \frac{b}{2\,(C + T)}\, [2\,T + 3\,C \pm \sqrt{(8\,T + 9\,C)\,C}]$$

If C equals T, x will be $0.219224\, b$. Having determined the position of the neutral axis and substituting the value for x in formula (3), three distinct quantities are obtained. Since the resultant of the intensities in such a triangle finds its point of application on the line drawn from the apex to the center of the base in a perpendicular distance of one-half the height of the triangle from the base, it is easy to calculate the leverage of the couple. For instance, if $x = s_1$; $b - x = s_2$; $\frac{1}{6}\, C \frac{a}{b}\, (b - x)^2 = p$; $\frac{1}{2}\, T\, a\, x = p_1$; $\frac{1}{6}\, T\, \frac{a}{b}\, x^2 = p_2$; then the moment of resistance will be : $p\, s = \frac{1}{2}\, p\, s_2 + (\frac{2}{3}\, p_1 - \frac{1}{2}\, p_2) s_1$ in which $p_1 - p_2 = p$ and $\dfrac{\frac{2}{3}\, p_1 - \frac{2}{3}\, p_2}{p_1 - p_2}\, s_1 = s_0$ or $p\, s = p\, (\frac{1}{2}\, s_2 + s_0)$; s_0 and $\frac{1}{2}\, s_2$ are measured from the neutral axis, and they are dependent on the position of this axis or on C and T.

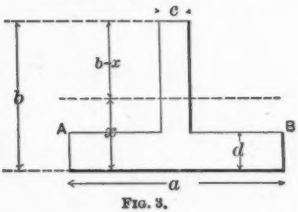


FIG. 3.

The neutral axis in a beam of the section shown in Fig. 3 may pass through the web, or through the flange, or through the edge $A\ B$ of the flanges, i. e., the sum of the intensities in the web may be greater, smaller, or equal to the intensities in the flange. Only the first two cases will be considered, as the third case needs no explanation.

First Case.—Let the neutral axis pass through the web of the beam, and let C be the crushing stress per square inch at the top of web, and T the tensile stress at the bottom edge of the flange; then the unit

of tensile resistance at $A\,B$ will be: $T\frac{x-d}{x}$, and the following equation for x may be written:

$$\tfrac{1}{2}\,C\,c\,(b-x) = \tfrac{1}{2}\,T\,a\,x - \tfrac{1}{2}\,T\,(a-c)\,\frac{(x-d)^2}{x} \ldots\ldots\ldots (5)$$

$$\text{or } x^2 + \frac{2\,T\,(a-c)\,d - C\,b\,c}{c\,(C+T)}\,x - \frac{T\,(a-c)}{c\,(C+T)}\,d^2 = 0$$

Second Case.—Let Fig. 4 represent the beam section when the neutral axis passes through the flange. If C denotes the crushing stress per square inch at the top edge of the flange, and T the tensile stress at the bottom edge of the web, then

$$\tfrac{1}{2}\,C\,a\,(b-x) = \tfrac{1}{2}\,T\,c\,x + \tfrac{1}{2}\,T\,(a-c)\,\frac{[x-(b-d)]^2}{x} \ldots\ldots\ldots (6)$$

$$\text{or } x^2 - \frac{C\,a\,b + 2\,T\,(a-c)\,(b-d)}{a\,(C+T)}\,x + \frac{T\,(a-c)\,(b-d)^2}{a\,(C+T)} = 0$$

The determination of the couple leverage s is similar to that in the case of the triangular shape, noting only that for each part of the beam the sum of intensities acts at two-thirds of their heights from the axis. In a similar way, x and s may be found for any shape.

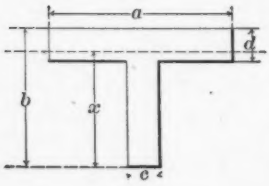


FIG. 4.

Table No. 1 has been prepared according to this theory. Three test pieces for each experiment have been cast from ordinary foundry iron, a mixture of about 50% 2 X Low Moor pig, and about 50% stove and machine scrap. Each specimen with a corresponding mark in column 2 was cast from the same ladle. All the specimens were broken in a 100 000-lb. Riehle testing machine, and tensile and crushing specimens prepared from the broken pieces. For the tensile tests the rectangular and triangular bars were turned down to about $\frac{7}{8}$ in. diameter for a distance of 6 ins., and from the T bars flats were cut about 0.4 x 0.8 ins. for a distance of 6 ins. For the crushing tests, specimens of about $\frac{3}{4}$ x $\frac{3}{4}$-in. base and $1\frac{1}{2}$ ins. high were cut from the rectangular and triangular bars; and pieces of about 0.4 x 0.4 in. base and about 0.8 in. high from the T bars.

While these tests are not all that is to be desired, they nevertheless serve to confirm the theory here presented. It at least clears up to the student of mechanics the difficulties of the moment of resist-

ance and shows that this moment is not always the same in the same beam in regard to the neutral axis. In many text books it is stated that to find the moment of resistance of an unsymmetrical section, divide the moment of inertia for safety by the greater distance of extreme fibers from the neutral axis, implying that the smaller distance would be also correct. This is clearly not true, because from the manner of derivation of the moment of inertia the greater value must be used, and the value so found multiplied by the stress coefficient.

In column 8 of Table No. 1 are given the center breaking loads as found by tests and by calculations, according to the theory here advanced. They correspond fairly well with each other except in the T bar section with the web up, where undoubtedly the wrinkling and buckling stresses added materially to the destruction of the material. The longitudinal and vertical shearing stresses were also not taken into account in these calculations. From equations (1), (3), (5) and (6) it is readily seen that the value of x or the position of the neutral axis does not change if multiples of T and C are taken, showing that these formulas are also correct in case the working stresses are taken in place of C and T. Equation (2) shows also that having two of the three values, C, T, and W, the third can be determined. For instance, a 1-in. square bar of brass, 12 ins. between supports, broke with a center load of 1 150 lbs. The tensile strength of this metal was found to be 17 970 lbs. per square inch. Substituting these values for T and W in equation (2), the value of C is found to be 24 408 lbs. per square inch.

This theory not only gives results more nearly in accord with those found by experiment than the common theory, but gives also a conception of the stresses working in a beam. In designing, it permits the choice of the most appropriate cross-section and material for any given purpose on a rational basis. In closing the author may add that he hopes a better and more extensive series of experiments bearing on this subject will be made on materials more homogeneous than those used for illustration in this paper.

DISCUSSION.

Mr. Marstrand. O. J. MARSTRAND, M. Am. Soc. C. E.—In referring to the common theory of flexure the author states only one of the fundamental laws upon which this theory is formulated, viz., that the intensity of fiber resistance is directly proportional to the distance from the neutral axis, and omits the other, viz., that under small stresses the changes of shape are approximately directly proportional to the forces that produce them; it is these two laws in conjunction that make the stresses above and below the neutral axis directly proportional to their distance. That the sum of the horizontal forces above and below the neutral axis must be zero is a fundamental law of pure mechanics, which is no more applicable to the theory of flexure than to any other. The author omits to state that the common theory of flexure is only supposed to be applicable to stresses within the elastic limit of the material for the very reason that the law referred to, the proportionality of stress to change of shape, or the value of E, is only assumed to be valid for small stresses. The word "small" is of course a relative term, having a different meaning for different materials; for wrought iron it is a much larger quantity than for cast iron and timber. In the latter cases it has been difficult to determine the elastic limit, but for the stresses used in designing in any of these materials, the common theory of flexure is generally accepted as rational. In order to substitute a new theory for the common theory, the author would have to prove, first, that the common theory is not rational within the limit of elasticity, for it is within this limit only that the designer is working, and this the author does not attempt to do at all; secondly, he would have to prove that the theory which he does offer is rational within the elastic limit, and this does not follow absolutely from his assumption and experiments made for rupture only. It is quite true, as he states, that if any multiple of C or T be taken, the value of x in equation (1)* remains the same, but it does not follow that C and T, assumed as maxima for rupture, should decrease in the same ratio, especially when the stresses range below the elastic limit.

Referring to the assumption mentioned, upon which the author bases his theory, the speaker questioned its general applicability, even to cases of rupture, and could not accept, without further argument at least, the hypothesis that the resistances of the lower and upper fibers should reach their maxima simultaneously, no matter what the shape of the section. Neither could he accept the theory that stresses beyond the elastic limit decrease in direct proportion as

* See page 486.

they approach the point of rotation. Referring to the experiments made by the author, it is unfortunate that the discrepancies between the actual and computed stresses should be greatest in the case of the T-shape cast-iron beam, with its flange down, as this is in a modified form, the proper shape to be used for such beams. The point is well taken, that the buckling of the web led to the irregularities, as it shows the importance of having also a small compression flange in such beams. Mr. Marstrand.

H. W. BRINCKERHOFF, M. Am. Soc. C. E.—According to the assumptions of the author any beam would be as likely to fail by compression as by tension, because the neutral axis so adjusts itself that the extreme fibers on each side are just at the point of rupture at the same time. In fact, all beams should theoretically fail on both sides at once, which does not agree with observed results. It is also difficult to see how, upon his theory, anything is gained by using a T section for cast-iron beams, which is found in practice to be a good form to employ if the rib is in compression. Mr. Brinckerhoff.

W. H. BREITHAUPT, M. Am. Soc. C. E.—Investigations into the theory of flexure, to be of any value, must be confined to what takes place below the elastic limit. It has been long established that the ratio of stress to strain is constant below this limit, while beyond it this does not hold true. It is therefore futile to attempt by means of observation of what takes place at rupture to establish something governing conditions below the elastic limit. The author's table of results of tests, given in support of his theory, indicates in the large variation in the results of experiments on pieces of similar size the difficulty of securing comparable data, even with the greatest care, with a material so lacking in uniformity as cast iron. Mr. Breithaupt

There is a considerable variation of most of the actual results from those computed on the author's theory. In the group where approximate agreement might be looked for, that of an inverted T section giving compression in the smaller area and tension in the greater, the variations are greatest, being as much at 20.2% in one case. Applying the common formula for rupture,

$$R = \frac{0.75\ (2\ P + W)\ l}{b\ d^2},$$

to the rectangular specimens, the first six of the table, obtaining from the sixth the value of $R = 28\ 140$, and using this for the other five, the results obtained for breaking loads are, in the order of the table, 1 272, 1 266, 1 274, 2 470 and 2 501. Although these do not prove much, owing to the unreliability of the material, they are in every case in better accord with the actual results than those obtained by means of the author's theory.

Mr. Breithaupt. The term "moment of resistance" is used in the paper for the moment of inertia divided by the distance to the extreme fiber, $I \div n$. The term has another well-established meaning, $If \div n$. It has been proposed to call $I \div n$ the section modulus and designate it by the symbol S, as is done by Prof. Reuleaux, and this might well be generally adopted.

Mr. Emery. CHARLES E. EMERY, M. Am. Soc. C. E.—The entire discussion has proceeded on the assumption that the longitudinal stresses upon the particles of material at any vertical cross-section of a beam are proportional to the distances of the particles from the neutral axis; whereas it can be proved easily that this assumption is incorrect, as is shown, moreover, by the behavior of beams. In practice a beam will not break until, on this assumption, the theoretical stress upon the outer fibers is double or at least much in excess of that which the material will resist in the testing machine, showing plainly that the portion of the material near the neutral axis carries a larger proportion of the load than its position would indicate according to the common theory. The reasons why this is possible were given by the speaker 17 years ago in a paper,* with illustrations, entitled "Flexure and Transverse Resistance of Beams."

Mr. Waite. GUY B. WAITE, Assoc. M. Am. Soc. C. E.—Mr. Emery's theory shows an accommodation in the distribution of stresses and strains produced during the transverse loading of beams, which is not recognized by the common theory of flexure, while the author's theory shows an accommodation in the location of the neutral axis under such a condition. Both theories give reasonable explanations of the action of materials having unequal tensile and compressive capacities at the point of rupture. It is generally considered that the extreme fiber strains at the point of rupture of rectangular cast-iron beams, as computed according to the usual theory, are about twice the direct tensile strength of the material. Mr. Emery's explanation of the discrepancy is that the distribution of stress through the material is such that the inner fibers receive more than their relative share according to the usual theory, and the author's explanation is that the neutral axis assumes a location giving, in the case of cast-iron sections, a materially increased area to withstand the tensile stresses. Whether or not these theories hold true below the elastic limit, they seem worth consideration when studying the action of materials at the point of rupture in cross-breaking. It is not improbable that, as the adjustment between stresses and strains takes place according to Mr. Emery's theory, the neutral axis is shifted in the manner outlined by the author, so that a complete theory of cross-breaking should take account of both these actions.

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* See *Transactions*, Vol. VIII, p. 149.

CORRESPONDENCE.

Mr. Fortin. S. J. FORTIN, Assoc. M. Am. Soc. C. E.—There are many, even to-day, who think that the present theory of flexure in beams is incomplete owing to the variations of the coefficient of elasticity E within and beyond the elastic limits. In all American books the writer has examined, E has always been defined as the ratio of unit stress to unit deformation; in algebraical language, denoting by s the unit stress, by λ the unit deformation, this definition would read: $E = S \div \lambda$. If experiments are made on a bar of length l, of section A, and with a load P, the result would be:

$$E = \frac{P}{A} \div \frac{\lambda}{l} = \frac{P\,l}{A\,\lambda} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Another method has been suggested by European writers which is claimed to give closer results, at the same time giving for E a more constant value. It assumes that the increase of deformation $d\,\lambda$ produced by an increment of load $d\,P$ is proportional, not to the original length, l, of the bar, but to the actual length, $l + \lambda$, hence:

$$\frac{d\,P}{A} = E\,\frac{d\,\lambda}{l + \lambda}.$$

By integrating, the following expression is obtained:

$$\frac{P}{A} = E\,[e\,(l + \lambda) + C],$$

in which e is the base of the Naperian logarithmic system:

When $P = O$

$$O = E\,[e\,(l) + C]$$
$$C = -\,E\,e\,(l)$$

therefore

$$\frac{P}{A} = E\,[e\,(l + \lambda) - L\,(l)],$$

and

$$\frac{P}{A} = E\,e\left(\frac{l + \lambda}{l}\right) \ldots\ldots\ldots\ldots\ldots (2)$$

Developing by series:

$$e\left(\frac{l + \lambda}{l}\right) = \frac{\lambda}{l} - \frac{1}{2}\left(\frac{\lambda}{l}\right)^2 + \frac{1}{3}\left(\frac{\lambda}{l}\right)^3 - \frac{1}{4}\left(\frac{\lambda}{l}\right)^4 +, \text{ etc.}$$

Neglecting the factors squared, cubed, etc.,

$$\frac{P}{A} = E\,\frac{\lambda}{l},$$

or

$$E = \frac{P\,l}{A\,\lambda},$$

which is identical with equation (1).

Mr. Fortin. It has always been found that the deformations increase in a quicker ratio than the loads, and that therefore the coefficient E decreases as the unit stress increases. The law regulating the decrease of E is not the same for all bodies. Thus in experimenting on the extensions of bars of different materials tested to rupture, it has been found that for some materials, such as cast iron and cast steel, the decrease of E was slow and regular from the beginning of the experiment to rupture, in such a manner that if the successive unit stresses are plotted as abscissas, and the corresponding values of E as ordinates a curve $a\,b$ (Fig. 6) is obtained which might be called a curve of elasticity, whose minimum ordinate $E\,t$ (value of E at rupture) is a considerable fraction of $E\,o$ (value of E at the origin of stress).

For other materials, such as wrought iron, soft steel, lead and copper, the decrease of E presents two phases very distinct; the first regular and slow as for cast iron; then a considerable fall is recorded, after which the decrease is again regular and slow until rupture. The curve of elasticity will then appear as $a'\,b'$ (Fig. 7), in which $E\,t$ (value of E at rupture) is only a small fraction of $E\,o$ (value of E at origin of stress).

FIG. 6.

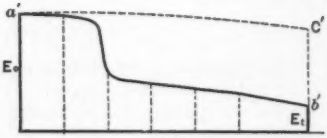

FIG. 7.

In Figs. 6 and 7 the curves $a\,b$ and $a'\,b'$ respectively are the graphical representations of formula (1), $E = P\,l \div A\,\lambda$, in which the deformations are proportional to the original length l of the bar, while in the same figures the dotted curves $a\,c$ and $a'\,c'$, respectively, are the graphical representations of formula (2),

$$\frac{P}{A} = E\left[L\left(\frac{l+\lambda}{l}\right)\right]$$

in which the deformations are proportional to the actual length $(l + \lambda)$ of the bar. The curves $a\,c$ and $a'\,c'$ decrease more slowly than $a\,b$ and $a'\,b'$; moreover $a\,c$ and $a'\,c'$ are similar in shape and equation.

In making experiments, the length of the specimen must be taken into account, as materials do not stretch evenly the total length of the specimen. It is true that the area of the cross-section should not be disregarded, as is the custom at present. It is easy to conceive that after stretching 25% the cross-section of the bar is notably reduced.

Mr. DuBois. A. J. DuBois, M. Am. Soc. C. E.—The author starts with the two assumptions that the lower and upper fibers of beams under a transverse breaking load are strained simultaneously and at the same section to their maximum fiber resistances T and C respectively, and that

the force in any fiber is directly proportional to its distance from the rotative axis. These assumptions seem very questionable. In accordance with them, the author deduces the position of the axis of rotation and the moment of the fiber forces. He thus obtains formulas which give the breaking load in terms of T and C. These formulas he then tests by a series of experiments. He concludes from the results of these experiments that the breaking loads given by his new formulas accord, not only fairly well with the actual breaking loads, but also that his theory gives better results than the common theory. Mr. DuBois.

This is important, if true, and worth examination. Before discussing, therefore, what seems questionable in his assumptions, it would be well to discuss these experiments. A glance at his table of results suggests at once certain points. It will be noticed that although the material was the same in all cases, yet the values of T and C vary considerably in each beam. In applying his formulas the author has the advantage of knowing and using in each case the values of T and C belonging to that case. While this is right in testing his formulas, yet in actual practice he would be obliged to employ a single average value of T and C for all the cases. The table gives no definite information as to how the results thus obtained would compare with experiment, but evidently they would not agree so well. The table thus gives a better showing for the new formulas than would be given by their practical use. The application of this point will be seen later.

The results obtained by the common theory are not given in the table. It is not shown by direct comparison that the results of the new formulas are better than those given by the formula in common use. As this is the entire point at issue, the omission is unfortunate, and the writer computed the strength of some of the beams by the common formula. Time has not allowed him to compute the results for the T bars, but there is no doubt that the same conclusions would hold for them as for the others. The common formula is $\frac{P\,l}{4} = \frac{R\,I}{v}$ where P is the breaking load, l the span, I the moment of inertia of the cross-section with reference to an axis through the center of gravity, v is the extreme fiber distance on the breaking side, and R is the strength of this extreme fiber. From the first experiment R is found to be 28 710 lbs. per square inch, which was used in all the computations. Evidently R might have been found for each case and the average result taken, which would give a better showing for the common method. It has been pointed out previously that the values of T and C in each case have been used in the new formulas, instead of an average value of T and C for all the cases, and this advantage is also on the side of the new formulas. These two points in favor of the new formulas are to be kept in mind while referring to the following table, in which there is given for each experiment the actual strength, the strength

Mr. DuBois. by the new theory and its ratio to the actual, the strength by the common theory and its ratio to the actual, and finally for each case the advantage of each method.

Series.	Experiment.	Strength by experiment.	Strength by new theory.	Ratio to actual strength. Percentage.	Strength by common theory.	Ratio to actual strength. Percentage.	New theory advantage. Percentage.	Common theory advantage. Percentage.
1	I.....	1 316	1 450	110.1	1 316	100.0	.....	10.1
	II.....	1 165	1 323	113.6	1 263	108.4	.....	5.2
	III.....	1 446	1 585	109.6	1 317	91.0	.....	0.6
2	I.....	2 400	2 535	105.6	2 538	105.7	0.1	.....
	II.....	2 321	2 519	108.5	2 579	111.1	2.6	.....
	III.....	2 501	2 429	97.1	2 575	102.9	0.0	0.0
3	I.....	1 346	1 364	101.3	1 342	99.7	.....	1.0
	II.....	1 386	1 348	97.2	1 334	96.2	1.0	.....
	III.....	1 725	1 640	95.0	1 421	82.3	12.7	.....
4	I.....	665	760	114.3	630	94.7	.....	9.0
	II.....	826	969	117.3	710	85.9	.....	3.2
	III.....	860	962	111.8	680	79.0	9.2	.....
						Total.....	25.6	29.1

The tabular comparison shows that the results given by the new theory are not so nearly in accord with fact as the results given by the common theory, even although the value for R is not an average, but is deduced from the first experiment only, and although the values of T and C for each case have been used in the new formulas, instead of an average value of T and C for all the cases. If average values for R, T and C had been taken, the comparison would be still more unfavorable for the new theory.

Case III of the third series, which gives the new theory 12.7% advantage, or about half of the total, is anomalous. The beam broke in two places 4½ ins. from the center. If this case is neglected, the common theory comes out far ahead, even with the handicaps mentioned. In addition to this, it gives a formula much more simple and readily applied. With this as the outcome of the experimental results it does not seem necessary to discuss the fundamental assumptions upon which this new theory is based.

Mr. Brown. CHARLES O. BROWN, M. Am. Soc. C. E.—The theory of bending cannot be applied in the form the author uses it for stresses beyond the elastic limit, or for materials following no defined laws of elasticity, such as cast iron. Writers on the subject, who credit cast iron with a defined limit of elasticity, place the difference of stress at that limit as two to one, respectively, for compression and tension.

As structures are designed for the purpose of supporting loads without causing permanent change of form, it would be preferable to introduce in the demonstration the difference between compression and tensile resistance at the limit of elasticity. Taking one of the Water-

town Arsenal tests on cast-iron columns, the writer finds a permanent set with a stress of about 3 000 lbs. per square inch, and a modulus of elasticity of 17 600 000 lbs.; in the same test with a stress of 16 000 lbs. per square inch, the modulus of elasticity is computed to be 13 000 000 lbs. This shows what reliance may be placed upon the laws of elasticity when applied to cast iron. Mr. Brown.

When beginning the work on the revision of the building laws for the City of New York, the writer intended to treat cast iron, stone, concrete and other materials not having a defined elastic limit by computing the resisting moments with reference to the extreme compression fiber, and using in this computation a low fiber stress for tension, about 2 000 lbs. for cast iron. This empirical rule would give satisfactory results for members of rectangular section; for designing other shapes, general specifications should guard against mistakes. For an illustration, in a cast-iron beam of the common form of a ⊥ lintel, not more of the width of the bed than would equal one-half the height of the rib should be considered in the computation, and the width of the rib on the top should be made at least equal to one-twentieth of the span.

The writer agrees fully with the author that the neutral axis does not always go through the center of gravity of the section, as insisted upon by most authors, and he has made use of this fact for many years, especially in designing tension and compression members subjected to bending strains in addition to their direct strains. This matter was fully discussed while considering the subject of eccentric loading of columns at the recent meetings for the revision of the building laws.

W. E. Belknap, Assoc. M. Am. Soc. C. E.—Mention was made in the discussion of the paper of the appearance of fractured beams, and conclusions were drawn directly therefrom as to the portions of the section in tension and in compression at the moment of rupture. The writer believes that a fractured specimen has no trustworthy superficial record of the condition of its fibers at that time. It is clear that as soon as the first fibers have given way the original working section has been destroyed, a new and smaller section being the result, and this before actual severance or destruction of the fibers. As fracture continues the working section diminishes, the neutral axis necessarily moving away from the side on which fracture began, and fibers which were in compression at the moment of giving way show at cessation of fracture unmistakable signs of having been finally in tension, or *vice versa*.

It seems natural that experience should show a larger number of fractured specimens to have given way on the tension side than on the compression side. It may be accounted for by the fact that, as the fibers yield on the compression side, fibers back of them are brought into action, while on the tension side this cannot take place, and each

Mr. Belknap. fiber in the line of fracture acts alone. While at the beginning of distortion or deflection the neutral axis may move toward the tension side by reason of the giving way of the exterior fibers in compression, it may reach eventually a point where there is an equilibrium of the forces in compression and tension. Then, when more strain is added, it may move back again as the tension fibers give way, and finally, when fracture has ceased, lie far within the compression side of the original section.

Mr. Prichard. HENRY S. PRICHARD, M. Am. Soc. C. E.—The author calls attention to the fact, well known among those who have to deal with the strength of materials, that the transverse breaking loads for beams cannot be determined with any certainty by methods based on the ordinary assumptions of flexure. This fact is in strict accordance with theory and not in conflict with it, as some suppose, as it is a necessary consequence that rules founded and dependent on the assumption that the elastic limit has not been exceeded are not applicable, except empirically, to cases in which the elastic limit has been exceeded.

The author, in discussing the subject of breaking loads, states in general that the results of "the theory of the transverse strength of beams" are unsatisfactory. He has not explicitly stated that he means this conclusion to apply to the results given by the theory in dealing with working stresses, but as he has given a theory of his own, in conflict with the commonly accepted one, and as he claims that his theory is correct, both for ultimate and working stresses, it must be inferred that he intends his criticism to apply to cases in which the strains are less than the elastic limit as well as to those in which they are in excess.

He bases his own theory on two assumptions:

First.—That the upper and lower fibers of beams under transverse breaking load are strained to their maximum fiber resistances.

Second.—That the stresses diminish in intensity, from the extreme fibers to the rotation axis, where they become zero, in direct ratio to the distances of the fibers from the axis.

The author has not given any demonstration to prove the theoretical accuracy of these assumptions, but seems to rely on certain observed phenomena and on the comparison of the actual breaking loads in given tests with figured breaking loads for the substantiation of his theory.

The phenomena cited admit of another explanation, and in one instance indicate a conflict with, instead of a confirmation of, his theory. This instance is the fact that "in breaking timber beams, the fibers in tension rupture before the compression fibers are affected, although the tensile strength of the wood may be four or five times as great as its compression strength." This fact certainly indicates that the rules founded on the ordinary assumptions of flexure are not applicable to

the determination of the breaking loads of the timber beams in question, but it also shows that in these beams the upper and lower fibers were not both strained to the maximum fiber resistances as his theory assumes. Mr. Prichard

The comparison of the actual breaking loads with figured ones shows considerable variation between them. To aid in the comparison the writer has calculated the percentage of error in the figured results and gives them in following statement :

Section.	First, Rectangular.			Second, Rectangular.			Third, Triangular.		
	I.	II.	III.	I.	II.	III.	I.	II.	III.
Breaking load, test	1 316	1 165	1 446	2 400	2 321	2 501	1 346	1 386	1 725
Breaking load, rule	1 450	1 323	1 585	2 535	2 519	2 429	1 364	1 348	1 640
Percentage difference	+10.2	+13.6	+9.6	+5.6	+8.5	−2.9	+1.3	−2.7	−4.9

Section.	Fourth, Triangular.			Fifth, Tee.			Sixth, Tee.		
	I.	II.	III.	I.	II.	III.	I.	II.	III.
Breaking load, test	665	826	860	4 770	4 520	4 990	2 400	2 220	2 080
Breaking load, rule	760	969	962	5 822	5 839	5 975	2 441	2 042	1 826
Percentage difference	+14.3	+17.3	+11.9	+22.1	+29.2	+19 7	+1.7	−8.0	−12.2

It will be noticed that the errors range from 12.2% deficiency to 29.2% excess, giving a total range of error of 41.4%, and that the average error is 7.5% excess. The author's explanation, in regard to the cases giving the most unfavorable comparisons, to the effect that "the wrinkling and buckling stresses add materially to the destruction of the material" should be given due weight in considering the comparisons.

In the discussion of the ordinary theory and the development of his own, the author has omitted all reference to the relations which exist between distortions and strains. A reasonable assumption as to these relations should constitute the hypothesis upon which any theory of the strains produced by transverse forces should rest. The relations between the intensities of the strains in the different fibers

Mr. Prichard. at their intersections with any given transverse plane, and the position of the neutral axis, are not subjects for independent assumption, but are necessary consequences of the relations existing between strain and distortion. The assumptions made by the author are not consistent with any conception of these relations which it is reasonable to entertain.

Let $A\ C$ and $B\ D$, Fig. 5, represent two transverse sectional planes, which in the free state of the beam are parallel to each other, the plane $A\ C$ being located at the fixed end, and the plane $B\ D$ so near to it that the bending moment at that plane does not differ sensibly from that at the end. Under the influence of the load W, the material will distort, the beam will deflect and the transverse sectional planes will no longer be parallel, the fibers at $A\ B$ will lengthen, at $C\ D$ shorten, at some longitudinal surface indicated as $E\ F$ will remain unchanged, and the ratio between the longitudinal distortions at different distances from $E\ F$ will be directly proportional to such distances. The dis-

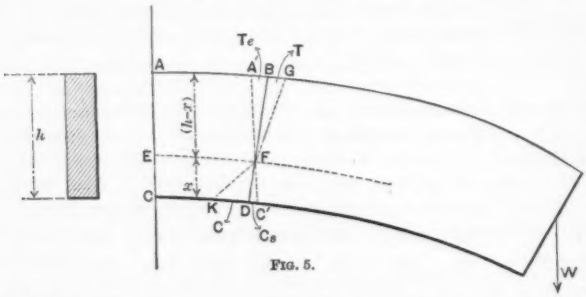

FIG. 5.

tance between the lines $B\ F\ D$ and $A'\ F\ C'$, parallel to $A\ E\ C$, at any given distance from F, is the distortion of the fibers at such a distance. As the fibers at $E\ F$ have neither been shortened or lengthened, the stress in these fibers must be zero and the point F therefore lies in what the author calls the axis of rotation.

From the author's first assumption let the lines $B\ G$ and $D\ K$ represent the maximum fiber resistance in tension and compression respectively.

Let e = the elongation of the fibers $A\ B$, for a tensile stress of 1 lb. per square inch.

Then Te = line $A'\ B$.

Let s = the shortening of the fibers $C\ D$ for a compressive stress of 1 lb. per square inch.

Then Cs = line $D\ C'$

$$Cs : x :: Te : (h - x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

$$Cs = \frac{x}{h - x}\, Te, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b)$$

Mr. Prichard.

From the author's equation—

$$\tfrac{1}{2}\, C\, a\, x = \tfrac{1}{2}\, T\, a\, (h - x)$$

$$\frac{x}{(h - x)} = \frac{T}{C} \ldots\ldots (c)$$

By substituting in equation (*b*)

$$C s = \frac{T}{C}\, T e \ldots\ldots (d)$$

$$s\, C^2 = e\, T^2 \ldots\ldots (e)$$

The writer can conceive of no reason why the relation indicated by equation (*e*) should exist. Substituting in equation (*e*) the values given by the author for cast iron, on page 486:

$$(100\,000)^2\, s = (20\,000)^2\, e \ldots\ldots (f)$$

$$e = 25\, s \ldots\ldots (g)$$

That is to say, the logical conclusion from the author's assumptions is that in rectangular cast-iron beams the elongation due to tension is 25 times as great per unit of stress as the shortening due to compression, and this conclusion applies, not only to ultimate stresses, but to any stresses, as is apparent from the second assumption, and also from the author's claim that his formulas are correct in the case of working stresses. In the light of present knowledge of cast iron, the assumptions which lead to such a conclusion regarding the relations between distortion and stresses within the elastic limit are not tenable.

From the author's second assumption and the fact that the distortions are directly proportional to their distances from the neutral axis, it follows that the distortion is directly proportional to the intensity of the stress. This conclusion is generally accepted as correct, provided the elastic limit is not exceeded.

At the moment of failure from a transverse load, the fibers at one or both extremes are strained beyond the elastic limit, while those near the neutral axis are not. It is well known that material strained beyond the elastic limit distorts more per unit of stress than material which is strained within such limit, consequently the author's second assumption is not a correct one on which to base a formula for determining breaking loads.

In the foregoing it has been shown that the author's theory is not correct in the case of either ultimate or working stresses. It may be, however, that with some modification to keep their results on the side of safety, his formulas can be used empirically to determine the breaking loads for certain kinds and shapes of beams, as his experiments seem to point that way.

A theoretically correct formula for breaking loads for beams would be an addition to general knowledge and have a scientific interest for engineers, but would not be of much use for determining working loads.

Mr. Prichard. A load which is but slightly in excess of the one which just strains the material to the elastic limit will work permanent injury to the beam, and, if long continued, will probably result in total failure, but it will take a considerably larger load to cause the beam to break immediately after its application. As the consequences of immediate breakage are more to be dreaded than those of permanent injury, the ratio in which the breaking load is required, for safety, to exceed the working load should be considerably greater than the ratio in which the load which will just strain the material to the elastic limit is required, for safety, to exceed the working load. Nevertheless, for beams, the relations between the breaking loads and the loads which will just strain the material to the elastic limit are such that, with the ratios which the judgment of most engineers dictates, it is the load which just strains the material to the elastic limit and not the breaking load, from which the working load is determined. In the discussion of working loads the writer has assumed that the compression flange is stayed against deflection sideways. When beams are not so stayed, buckling, as well as injury and breaking, has to be guarded against, and may become the controlling factor in determining the working load.

Mr. Johnson. J. B. JOHNSON,, M. Am. Soc. C. E.—While the writer has no objection to an effort to show a purely empirical relation between the ultimate strength of materials in tension, compression and cross-bending, he objects strongly to such an effort to place the relationship upon a theoretical basis as has been attempted in this paper. The ordinary theory of the strength of a beam is perfectly correct and entirely applicable to all materials within their elastic limits. It does not hold at rupture for any material which is not perfectly elastic up to rupture, neither does it make any pretense to such an application. Any amount of evidence, therefore, of the failure of the ordinary theory of cross-bending strength when applied at the breaking limit is not to the point. For all purposes for which the engineer has occasion to use these formulas in designing, that is to say, for the greatest working loads which should ever be allowed on any material, the ordinary theory is absolutely correct, and does give the actual stress on the extreme fibers for all such loads. The so-called modulus of rupture in cross-bending, therefore, is well known to be a myth, so far as it is supposed to represent any actual stress on the outer fibers at rupture. It is, however, a perfectly legitimate and convenient modulus to use in designing, since, when the factor of safety is applied, the formula becomes perfectly applicable.

The writer's particular objections to the position taken by the author are:

First.—He assumes without any show of reason whatever that when any beam is broken transversely, the stresses on the extreme fibers on

both the tension and the compression sides of the beam reach their breaking limits simultaneously. That is to say, he assumes that the compressive stress is at its breaking limit on the one side, when the tension stress reaches its limit on the other side. He offers no kind of evidence that this is correct. All persons familiar with the testing of materials in cross-bending are fully aware that this is very far from being correct. In timber, for instance, a beam usually crushes down on the compression side long before it fails on the tension side; whereas, in cast-iron the failure on the tension side occurs long before the material has reached its compressive strength on the opposite side. This assumption most effectually undermines and vitiates the theoretical equations and conclusions which rest upon it. Mr. Johnson.

Second.—In the derivation of formula (1), page 486, the author states that the total tensile stress on one side of the neutral axis must equal the total compressive stress on the other side. In this he is correct, and has not made the usual mistake of persons who object to the ordinary theory of cross-bending strength; but he has made in this connection another mistake which entirely vitiates his equations. He has assumed that the rate of variation of stress on the two sides of the axis is the same, after the neutral axis has shifted from its normal position, which is that of a plane through the center of gravity of the cross-section. That is to say, his a is the same on the two sides of his fundamental equation, which agrees with the stress line shown in his Fig. 1. His figure, however, shows correctly the effect of this assumption, in that it makes the stress area on the compression side of the beam toward which the neutral axis has moved very much less than the corresponding area on the tension side, this being a necessary result of the uniform variation of stress across the section and the shifting of the neutral axis. In other words, these stress areas cannot remain equal under such assumptions (although that they must always be equal is a purely mathematical or geometrical fact), and hence his fundamental equation cannot be true on the assumptions made. The only way in which that equation can be made true is to make the stress vary by a different law on the two sides of the section. As a matter of fact, after the neutral axis has begun to move from its normal position on one or both sides of this plane, the stress no longer varies uniformly, but varies in some one of a number of ways.*

It is well known that the cross-breaking modulus of any material at rupture lies somewhere between the ultimate tensile and compressive strengths, but no theory is competent to determine this relation, since it varies with every different kind of material, and is a function in each case of the particular characteristics of the strain diagrams of

* See "Modern Framed Structures," by Johnson, Bryan and Turneaure, p. 124. This relation is also shown clearly for cast iron in the writer's paper on "Cast-Iron—Strength, Resilience, Tests and Specifications," *Transactions*, Vol. XXII, p. 91.

Mr. Johnson. that material, in both tension and compression, and is mainly dependent on the form of the diagram of the weaker kind of stress. Any such formulas, therefore, as are derived in this paper must be regarded as resting on a purely empirical basis, and without any kind of theoretical foundation.

Mr. Lewinson. M. LEWINSON, M. Am. Soc. C. E.—The objections to this paper may be summarized as follows: First, that the formulas are empirical; second, that they do not apply within the limit of elasticity; third, that they give no better results than the common theory.

As to the first objection, if the author's deductions are empirical, why are not the formulas of the common theory also empirical? Is not the common theory based on the elasticity of material presumably derived from experiments? Prof. Johnson calls the modulus of rupture a myth, but nevertheless he calls the common theory perfectly correct. He states further that the author's assumption of the simultaneous maximum fiber stresses is without any show of reason whatever. To this the author would answer that long before he came to the conclusions advanced in the paper, he made many experiments, principally on natural sticks from which the bark had been removed carefully. These were broken transversely around a cushion to eliminate as much as possible any local secondary pressure at the breaking point. In more than 60% of such breakages the tension fibers ruptured before the compression fibers were affected. Of these again the majority split longitudinally in or at the tension fibers. In experimenting with square pieces of timber, similar phenomena were noticed, although in a smaller proportion. In experimenting with beams of brittle materials, he was unable to observe on which side the fibers broke first, as they broke suddenly. Cases are on record, however, where in a cast-iron beam of I section the compressed flange broke first.* On these results the author based his conclusions, and though the experiments were crude, still he thinks he was justified in doing so.

As to the second objection, any mention of elasticity of material was purposely avoided in the whole paper. At first the author showed in Fig. 1 the lines representing the stress area, but he changed it subsequently, as it seemed to him that such a figure would be confusing. He used the rotation plane without any attempt to show how the fibers are stretched or compressed, for the simple reason that he does not presume to know how this is done. He does assert, however, that all the recorded results of experiments on compression under increasing strains are incorrect. They all were made, so to speak, on columns the length of which was too great in comparison with the diameter, and consequently the amount of observed compression was apparent instead of real. For instance a bar 9 ins. long under compressive strain, free to move sidewise 0.01 in. in single flexure, will produce an apparent

* See "Resistance of Material," by DeVolson Wood.

shortening of 0.000003 in. for each inch of column. In double flexure it will produce 0.000013 in. for each inch. In triple flexure it will produce 0.00003 in. for each inch. In following carefully the Watertown experiments on cast iron for the year 1891, it is almost possible to see when a new flexure occurs. For instance up to 4 000 lbs. pressure per square inch no flexure occurred.* The first flexure started at 5 000 lbs., and continued up to 9 000 lbs. Between 9 000 and 10 000 lbs. a new flexure occurred, and at about 30 000 lbs. the triple flexure occurred, which remained until failure. Mr. Lewinson.

The law under which materials compress is not known exactly, and it is just as well to say that the amounts of compression and extension are in inverse proportion to the respective resistances, as to say that they are equal, even within the so-called limit of elasticity.

The author cannot agree with Prof. Johnson as to the shifting of the neutral axis. If it were true that in a cast-iron beam, under a transverse load, beyond the limits of elasticity, the fibers compress more than they extend under the same strains, the axis should shift toward the tensile fibers. Evidently this is not so. On the other hand, if a beam of a material whose resistance to crushing is infinite and whose resistance to tension is finite, is under a transverse load, the neutral axis would unquestionably pass through the outside compressed fibers. If the conditions are reversed, the neutral axis will pass through the outside tension fibers. If one of the two resistances is very great in relation to the other, it is only logical that the neutral axis will pass nearer those extreme fibers having the greater resistance.

As to the third objection, that the formulas give no better results than the common theory, so far as the author's experiments on cast iron show, this is true. Still it is an open question if experiments conducted with a more uniform and homogenous material would not make a better showing. Cast iron was used in the experiments for convenience only, and the author regrets that he was unable to experiment on bars of glass, as at first intended, because of the difficulty of obtaining the desired shapes in this material.

* See "Report of Tests of Metals, etc., at the Watertown Arsenal," 1891, p. 173.